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Hitting times for a correlated three dimensional brownian motion

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Abstract

In this paper we try to generalize the Iyengar's result in dimension 3. We prove that the method of images used in dimension 2 leads us to a tiling 3 dimensional space problem. A such a problem has only some particular solutions.

Keywords: three dimensional brownian motion, hitting time

1. Introduction

First passage time problems have been the subject of great interest in different fields of research such as micro-biology or finance. In a multivariate setting, computing analytical expressions for the distribution of hitting times is a very challenging task even in the case of low-dimensional correlated Brownian motions. In dimension 2, the result is due to Iyengar [3]. In this paper, we investigate the problem in dimension 3, i.e., we consider a 3-dimensional correlated Brownian motion starting from $x \in \mathbb{R}^3$ and we denote by $\tau_i = \inf\{t \geq 0 : B_t^i > a_i\}$ the first time its i -th components hits a constant barrier a_i . In some financial applications, computing the density of multivariate brownian motions before to exit the domain is of crucial importance. Muirhead [4] provides a general expression but tractable solutions are only obtained in dimension 2 and for a single particular case in dimension 3. The application of his work mainly concerns the pricing of multi-asset barrier options whose payoff is conditional on the trajectory of some underlying stock price processes and their ability to hit or avoid pre-specified barriers. In dimension 3, another example of application would be the assessment of bilateral counterparty risk for credit default swaps where default and spread risks of the three market participants (the protection buyer, the protection seller and the reference entity) could be described in a three-dimensional structural model. The CDS unilateral counterparty risk problem has been investigated by Blanchet-Scalliet and Patras [1] in a bivariate structural credit risk model. The distribution of the underlying process before exiting the domain is given as the solution of a heat equation which can be solved by using the method of image. In dimension 3, this problem is equivalent to find a way to fill the space with a trihedron. This question has a long and somehow complicated history. Cite (Sommerville et co). Therefore, it only exists few solutions to do this. In this paper, after recalling how to obtain the expression of density in the general setting, we show that the method of image used by Iyengar [3] in dimension 2 can only lead in dimension 3 to analytical expressions for 4 particular cases.

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2. Notations and Assumptions

Let $(X_t)_{t \geq 0}$ a three dimensional correlated Brownian motion starting from $(0, 0, 0)$ with $\mathbb{E}(X_t) = 0_{\mathbb{R}^3}$ and $Var(X_t) = t\Sigma$ where the correlation matrix is given by

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}.$$

We define the following stopping times $\tau_i = \inf\{t \geq 0 : X_t^i = a_i\}$, $a_i > 0$, where X_t^i is the i -th component of X_t , $i = 1, 2, 3$. Let $a = (a_1, a_2, a_3)$ and $\tau = \tau_1 \wedge \tau_2 \wedge \tau_3$.

Since Σ is a symmetric matrix, then Σ is diagonalizable. Let consider that the eigenvalues of the matrix Σ are $\lambda_1, \lambda_2, \lambda_3$. Let A be the matrix of the eigenvectors

$$A = \begin{pmatrix} v_1^1 & v_1^2 & v_1^3 \\ v_2^1 & v_2^2 & v_2^3 \\ v_3^1 & v_3^2 & v_3^3 \end{pmatrix}.$$

We also introduce the following matrix D such that

$$D^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{pmatrix}.$$

The matrix A is real orthogonal, so $A^{-1} = A'$ (where the transpose of matrix A is written A').

- $\det(\Sigma) \neq 0$

Let define $Z_t = D^{-1/2}A'(a - X_t)$. Then $(Z_t)_{t \geq 0}$ is a three dimensional uncorrelated brownian motion starting from $\hat{a} = D^{-1/2}A'a$. For $i \in \{1, 2, 3\}$ the stopping time τ_i is now the first hitting time of P_i by $(Z_t)_{t \geq 0}$ where

$$\begin{aligned} P1 &= \{(x, y, z) : v_1^1 \sqrt{\lambda_1} x + v_1^2 \sqrt{\lambda_2} y + v_1^3 \sqrt{\lambda_3} z = 0\}, \\ P2 &= \{(x, y, z) : v_2^1 \sqrt{\lambda_1} x + v_2^2 \sqrt{\lambda_2} y + v_2^3 \sqrt{\lambda_3} z = 0\}, \\ P3 &= \{(x, y, z) : v_3^1 \sqrt{\lambda_1} x + v_3^2 \sqrt{\lambda_2} y + v_3^3 \sqrt{\lambda_3} z = 0\}. \end{aligned}$$

Remark that

$$\cos(\widehat{P1, P2}) = \frac{(ADA')_{12}}{\sqrt{(ADA')_{11}(ADA')_{22}}} = |\rho_{12}|$$

where $(M)_{ij}$ is the (i, j) entry of the matrix M . Similary one has $\cos(\widehat{P2, P3}) = |\rho_{13}|$, $\cos(\widehat{P3, P1}) = |\rho_{23}|$.

Let W be the boundary domain delimited by $P1, P2$ and $P3$ and such as $\hat{a} = (x_0, y_0, z_0) \in W$, $W = \{(x, y, z) \text{ such that } \text{sgn}(P_i(x_0, y_0, z_0)) = \text{sgn}(P_i(x, y, z)), \ i = 1, 2, 3\}$. Remark that W is a trihedron and τ is the first hitting time of ∂W by $(Z_t)_{t \geq 0}$.

- $\det(\Sigma) = 0$

Suppose that a single eigenvalues of Σ is null (say λ_1). Then $P1 \cap P2 \cap P3 = (Ox)$. In this case W is not trihedron, it is a domain delimited by two of the three planes (the two planes are choosen according to the starting point position).

Remark that if two eigenvalues are null, then W is a half-space.

3. Density of the Brownian motion in the domain W

Let f a positive bounded function defined on W , and which vanishes on ∂W , then the function

$$u(t, \mathbf{x}) = \mathbb{E}^{\mathbf{x}} f(Z_{t \wedge \tau}) = \int_W f(y) \mathbb{P}^{\mathbf{x}}(\tau > t, Z_t \in dy)$$

satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u \text{ in } W, \quad u(0, \mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in W, \quad u(t, \mathbf{x}) = 0, \mathbf{x} \in \partial W. \quad (1)$$

Suppose that it is possible to tile the three-dimensional Euclidean space using our domain W only by plane symmetry. As in [3], we can solve (1) by the method of images. We denote by $S_k, k = 0, \dots, K$ the sequence of the symmetries. Let $T_0 = I_3, T_k = S_k \circ T_{k-1}$ and $\tilde{f}(\mathbf{y}) = (-1)^k f(T_k^{-1}(\mathbf{y})), \mathbf{y} \in T_k(W)$ Then our initial problem is to solve

$$\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2} \Delta \tilde{u} \text{ in } \mathbb{R}^3, \quad \tilde{u}(0, \mathbf{x}) = \tilde{f}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3.$$

Using classical technique of Fourier transform, the solution is

$$\tilde{u}(t, \mathbf{x}) = \int_W f(\mathbf{y}) \sum_{k=0}^K (-1)^k \frac{1}{(2\pi t)^{3/2}} \exp\left(-\frac{(\mathbf{x} - T_k(\mathbf{y}))(\mathbf{x} - T_k(\mathbf{y}))'}{2t}\right) d\mathbf{y}.$$

Then, it follows that

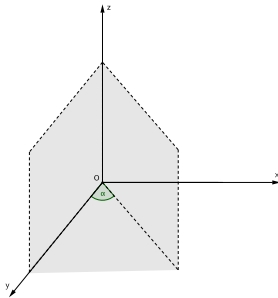
$$\mathbb{P}^{\mathbf{x}}(\tau > t, Z_t \in d\mathbf{y}) = \frac{1}{(2\pi t)^{3/2}} \sum_{k=0}^K (-1)^k \exp\left(-\frac{(\mathbf{x} - T_k(\mathbf{y}))(\mathbf{x} - T_k(\mathbf{y}))'}{2t}\right) d\mathbf{y}. \quad (2)$$

A similar formula is obtained in [4], but Muirhead writes it in dimension n . Even if this result seems to be very general, it is difficult to use it in practice. Indeed, to obtain an explicit solution, we need to identify the sequence of symmetries and all symmetric domains (Muirhead gives a single example). The problem is now a space-filling trihedra. It is not even easy to decompose \mathbb{R}^3 into symmetric trihedra.

4. Simple cases

We give here the explicit solution of some simple cases.

- *Correlation coefficients* $(\rho, 0, 0)$



Here our domain W (trihedron) can be delimited by the planes $P_1 : x = 0$, $P_2 : z = 0$ and $P_3 : \cos(\alpha)x + \sin(\alpha)y = 0$, where $\cos(\alpha) = \rho$. Let consider now that the departure point $\hat{a} \in W$.

First suppose that $\alpha = \frac{\pi}{m}$. Using the cylindrical coordinates it is easy to write the $4m$ -symmetries that we need to fill the space. Let $y = (r, \theta, z)$, then the sequence of the images of y is $T_k(y) = (r, \theta_k, z)$ where

$$\theta_{2k} = 2k\alpha + \theta, \quad \theta_{2k+1} = (2k+2)\alpha - \theta \quad \text{if } 0 \leq k \leq m-1$$

and $T_{m+k}(y) = (r, \tilde{\theta}_k, -z)$ if $0 \leq k \leq 2m-1$ where

$$\tilde{\theta}_0 = \theta_{2m-1}, \quad \tilde{\theta}_k = \theta_{k-1}, \quad \text{if } 0 \leq k \leq 2m-2.$$

Hence

$$\begin{aligned} \mathbb{P}^{(r_0, \theta_0, z_0)}(\tau' > t, Z_t \in (dr, d\theta, dz)) = \\ \frac{1}{(2\pi t)^{3/2}} e^{-\frac{r_0^2 + r^2}{2t}} \left(e^{-\frac{(z_0 - z)^2}{2t}} \sum_{k=0}^{2m-1} (-1)^k e^{\frac{r_0 r}{t} \cos(\theta_0 - \theta_k)} - e^{-\frac{(z_0 + z)^2}{2t}} \sum_{k=0}^{2m-1} (-1)^k e^{\frac{r_0 r}{t} \cos(\theta_0 - \theta_k)} \right) r dr d\theta dz. \end{aligned}$$

Using the same argument as in Iyengar [3], we obtain that

$$\begin{aligned} \mathbb{P}^{(r_0, \theta_0, z_0)}(\tau > t, Z_t \in (dr, d\theta, dz)) = \\ \frac{2r}{\sqrt{2\pi\alpha}t^{3/2}} \left(e^{-\frac{(z_0 - z)^2}{2t}} - e^{-\frac{(z_0 + z)^2}{2t}} \right) e^{-\frac{r_0^2 + r^2}{2t}} \sum_{n=0}^{\infty} (-1)^k \sin \frac{n\pi\theta}{\alpha} \sin \frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha} \left(\frac{rr_0}{t} \right) dr d\theta dz. \end{aligned}$$

We remark that the expression is still valid for any α .

This formula can be obtained directly using the Iyengar's result. Indeed as the Browian motion Z^3 is independant of (Z^1, Z^2) , one has

$$\mathbb{P}^{(r_0, \theta_0, z_0)}(\tau > t, Z_t \in (dr, d\theta, dz)) = \mathbb{P}^{(r_0, \theta_0)}(\tau_1 \wedge \tau_2 > t, (Z_t^1, Z_t^2) \in (dr, d\theta)) \mathbb{P}^{(z_0)}(\tau_3 > t, Z_t^3 \in dz).$$

• Null eigenvalues

If a single eigenvalue is null (say $\lambda_3 = 0$), then

$$\begin{aligned} \mathbb{P}^{(r_0, \theta_0, z_0)}(\tau > t, Z_t \in (dr, d\theta, dz)) = \\ \frac{2r}{\sqrt{2\pi\alpha}t^{3/2}} e^{-\frac{(z_0 - z)^2}{2t}} e^{-\frac{r_0^2 + r^2}{2t}} \sum_{n=0}^{\infty} (-1)^k \sin \frac{n\pi\theta}{\alpha} \sin \frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha} \left(\frac{rr_0}{t} \right) dr d\theta dz. \end{aligned}$$

If two eigenvalues are null ((say $\lambda_1 = \lambda_2 = 0$), then

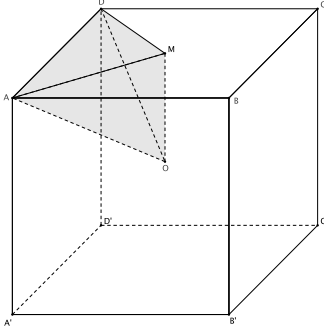
$$\mathbb{P}^{(x_0, y_0, z_0)}(\tau > t, Z_t \in (dx, dy, dz)) = \frac{1}{(2\pi t)^{3/2}} e^{-\frac{(x_0 - x)^2 + (y_0 - y)^2}{2t}} \left(e^{-\frac{(z_0 - z)^2}{2t}} - e^{-\frac{(z_0 + z)^2}{2t}} \right) dx dy dz.$$

5. Explicit calculus

Let imagine that the three-dimensional Euclidean space is reduced to a cube, then our domain W (trihedron) is reduced to a tetrahedron. The problem is now to find all symmetries which allow to our tetrahedron to fill the entire cube. Note that all the symmetric tetrahedra need to have a commun point and $S_0 * S_1 * \dots * S_K = I_3$. It is a particular problem of space-filling tetrahedra. Sommerville [5] discovered a list of exactly four tilings (up to isometry and re-scaling) and Edmonds [2] proves that this classification of tetrahedra that can tile the 3-dimensional space in a proper, face-to-face manner is completed. One of these four tilings needs two neighboring cubes for its construction and all the tetrahedra don't have a commun point. Another case doesn't allow to apply the method of images because the boundary conditions of (1) are not satisfied. Hence since, there exists only a few tetrahedra which can till the space, then we have only 2 particular cases where we can identify the explicit symmetries and compute the explicit form of (2).

We give the details for the first case, but a similar reasoning is used for the others.

- **Correlation coefficients** $(\frac{1}{2}, \frac{1}{2}, 0)$, $(-\frac{1}{2}, -\frac{1}{2}, 0)$ **or** $(\frac{1}{2}, -\frac{1}{2}, 0)$.

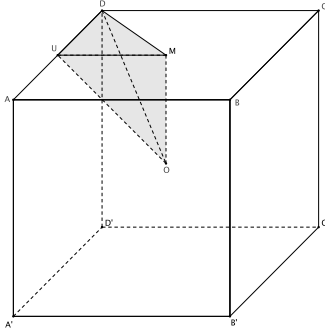


Let consider now that the departure point \hat{a} is in the tetrahedron $OADM$ where $M = \frac{1}{2}[BD]$. Let denote by S_1 the symmetry when the plane of symmetry is (ODB) , S_2 the symmetry when the plane of symmetry is (OAC) , S_3 the symmetry when the plane of symmetry is (OAD) and S_4 the symmetry when the plane of symmetry is (OCD) , S_5 the symmetry when the plane of symmetry is (OAB) and S_6 the symmetry when the plane of symmetry is (OCB) . By the following sequence of symmetries, the tetrahedron $OADM$ tills the entire cube and returns to the initial position: $S_1, S_2, S_6, S_5, S_4, S_5, S_2, S_6, S_3, S_6, S_5, S_2, S_1, S_2, S_6, S_5, S_4, S_5, S_2, S_6, S_3, S_6, S_5, S_2$.

In this case we have

$$\begin{aligned}
\frac{(2\pi t)^{3/2}}{dxdydz} \mathbb{P}^{(x_0, y_0, z_0)}(\tau' > t, Z_t \in (dx, dy, dz)) = & e^{-\frac{1}{2t}[(x_0-x)^2+(y_0-y)^2+(z_0-z)^2]} - e^{-\frac{1}{2t}[(x_0-y)^2+(y_0-x)^2+(z_0-x)^2]} \\
& + e^{-\frac{1}{2t}[(x_0+x)^2+(y_0+y)^2+(z_0-z)^2]} - e^{-\frac{1}{2t}[(x_0+x)^2+(y_0+z)^2+(z_0-y)^2]} + e^{-\frac{1}{2t}[(x_0+z)^2+(y_0+x)^2+(z_0-y)^2]} \\
& - e^{-\frac{1}{2t}[(x_0-x)^2+(y_0-z)^2+(z_0-y)^2]} + e^{-\frac{1}{2t}[(x_0-z)^2+(y_0-x)^2+(z_0-y)^2]} - e^{-\frac{1}{2t}[(x_0-z)^2+(y_0+y)^2+(z_0+x)^2]} \\
& + e^{-\frac{1}{2t}[(x_0-y)^2+(y_0+z)^2+(z_0+x)^2]} - e^{-\frac{1}{2t}[(x_0+z)^2+(y_0-y)^2+(z_0+x)^2]} + e^{-\frac{1}{2t}[(x_0+y)^2+(y_0-z)^2+(z_0+x)^2]} \\
& - e^{-\frac{1}{2t}[(x_0+y)^2+(y_0-x)^2+(z_0+z)^2]} + e^{-\frac{1}{2t}[(x_0-x)^2+(y_0+y)^2+(z_0+z)^2]} - e^{-\frac{1}{2t}[(x_0-y)^2+(y_0+x)^2+(z_0+z)^2]} \\
& + e^{-\frac{1}{2t}[(x_0+x)^2+(y_0-y)^2+(z_0+z)^2]} - e^{-\frac{1}{2t}[(x_0+x)^2+(y_0-z)^2+(z_0+y)^2]} + e^{-\frac{1}{2t}[(x_0+z)^2+(y_0-x)^2+(z_0+y)^2]} \\
& - e^{-\frac{1}{2t}[(x_0-x)^2+(y_0+z)^2+(z_0+y)^2]} + e^{-\frac{1}{2t}[(x_0-z)^2+(y_0+x)^2+(z_0+y)^2]} - e^{-\frac{1}{2t}[(x_0-z)^2+(y_0-y)^2+(z_0-x)^2]} \\
& + e^{-\frac{1}{2t}[(x_0-y)^2+(y_0-z)^2+(z_0-x)^2]} - e^{-\frac{1}{2t}[(x_0+z)^2+(y_0+y)^2+(z_0-x)^2]} + e^{-\frac{1}{2t}[(x_0+y)^2+(y_0+z)^2+(z_0-x)^2]} \\
& - e^{-\frac{1}{2t}[(x_0+y)^2+(y_0+x)^2+(z_0-z)^2]}.
\end{aligned}$$

• **Correlation coefficients** $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, 0\right), \left(-\frac{1}{2}, -\frac{\sqrt{2}}{2}, 0\right), \left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, 0\right)$ or $\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, 0\right)$



We consider the case where we cut the tetrahedron $OADM$ in half : so we suppose that the departure point \hat{a} is in $OAUM$ where $U = \frac{1}{2}[AD]$. This tetrahedron can still by 48 symmetries till the cube. Let introduce the following symmetries : S_1 the symmetry when the plane of symmetry is (OAD) , S_2 the symmetry when the plane of symmetry is (OCD) , S_3 the symmetry when the plane of symmetry is (OBB') and S_4 the symmetry when the plane of symmetry is (OZF) (where $Z = \frac{1}{2}[DD']$ and $F = \frac{1}{2}[CC']$) , S_5 the symmetry when the plane of symmetry is (OBC) , S_6 the symmetry when the plane of symmetry is (OAB) , S_7 the symmetry when the plane of symmetry is (OXI) (where $X = \frac{1}{2}[DC]$ and $I = \frac{1}{2}[D'C']$) , S_8 the symmetry when the plane of symmetry is (OAC) and S_9 the symmetry when the plane of symmetry is (OUV) where $V = \frac{1}{2}[BC]$. The sequences of the symmetries is $S_1, S_2, S_3, S_4, S_3, S_6, S_5, S_3, S_6, S_7, S_6, S_8, S_9, S_3, S_7, S_8, S_5, S_2, S_4, S_6, S_1, S_8, S_6, S_1, S_4, S_5, S_7, S_1, S_4, S_5, S_6, S_3, S_5, S_6, S_4, S_2, S_9, S_6, S_8, S_4, S_8, S_2, S_5, S_8, S_2, S_7, S_2, S_3$.

To find $\mathbb{P}^{(x_0, y_0, z_0)}(\tau' > t, Z_t \in (dx, dy, dz))$, it is enough to replace the sequence $T_k, k \in 0, \dots, 47$ in (2) by : $T_0(x, y, z) = (x, y, z)$, $T_1(x, y, z) = (x, -z, -y)$, $T_2(x, y, z) = (-z, x, -y)$, $T_3(x, y, z) = (-z, y, -x)$, $T_4(x, y, z) = (z, y, -x)$, $T_5(x, y, z) = (z, x, -y)$, $T_6(x, y, z) = (x, z, -y)$, $T_7(x, y, z) = (x, y, -z)$, $T_8(x, y, z) = (y, x, -z)$, $T_9(x, y, z) = (y, z, -x)$, $T_{10}(x, y, z) = (-y, z, -x)$, $T_{11}(x, y, z) = (-y, x, -z)$, $T_{12}(x, y, z) = (x, -y, -z)$, $T_{13}(x, y, z) = (-x, -y, -z)$, $T_{14}(x, y, z) = (-y, -x, -z)$, $T_{15}(x, y, z) = (y, -x, -z)$, $T_{16}(x, y, z) = (-x, y, -z)$, $T_{17}(x, y, z) = (-x, z, -y)$, $T_{18}(x, y, z) = (z, -x, -y)$, $T_{19}(x, y, z) = (-z, -x, -y)$, $T_{20}(x, y, z) = (-x, -z, -y)$, $T_{21}(x, y, z) = (-x, y, z)$, $T_{22}(x, y, z) = (y, -x, z)$, $T_{23}(x, y, z) = (y, -z, x)$, $T_{24}(x, y, z) = (-z, y, x)$, $T_{25}(x, y, z) = (z, y, x)$, $T_{26}(x, y, z) =$

(y, z, x) , $T_{27}(x, y, z) = (-y, z, x)$, $T_{28}(x, y, z) = (z, -y, x)$, $T_{29}(x, y, z) = (-z, -y, x)$, $T_{30}(x, y, z) = (-y, -z, x)$, $T_{31}(x, y, z) = (-y, -x, z)$, $T_{32}(x, y, z) = (-x, -y, z)$, $T_{33}(x, y, z) = (-x, -z, y)$, $T_{34}(x, y, z) = (-z, -x, y)$, $T_{35}(x, y, z) = (z, -x, y)$, $T_{36}(x, y, z) = (-x, z, y)$, $T_{37}(x, y, z) = (x, z, y)$, $T_{38}(x, y, z) = (z, x, y)$, $T_{39}(x, y, z) = (z, -y, -x)$, $T_{40}(x, y, z) = (-z, -y, -x)$, $T_{41}(x, y, z) = (-z, x, y)$, $T_{42}(x, y, z) = (x, -z, y)$, $T_{43}(x, y, z) = (x, -y, z)$, $T_{44}(x, y, z) = (-y, x, z)$, $T_{45}(x, y, z) = (-y, -z, -x)$, $T_{46}(x, y, z) = (y, -x, -z)$ and $T_{47}(x, y, z) = (y, x, z)$.

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